

PLANETARY WAVES ON BETA-PLANES

RICHARD D. LINDZEN

National Center for Atmospheric Research, Boulder, Colo.

ABSTRACT

The problem of linearized oscillations of the gaseous envelope of a rotating sphere (with periods in excess of a day) is considered using the β -plane approximation. Two particular β -planes are used—one centered at the equator, the other at a middle latitude. Both forced and free oscillations are considered. With both β -planes it is possible to approximate known solutions on a sphere. The use of either β -plane alone, however, results in an inadequate description. In particular it is shown that the equatorial β -plane provides good approximations to the positive equivalent depths of the solar diurnal oscillation, while the midlatitude β -plane provides good approximations to the negative equivalent depths. The two β -planes are also used to describe Rossby-Haurwitz waves on rapidly rotating planets, and the vertical propagatability of planetary waves with periods of a day or longer.

1. INTRODUCTION

One of the simplest general types of problems of importance to atmospheric dynamics is that of linearized wave motions in the gaseous envelope of a rotating sphere. The waves are generally taken to be small perturbations on a barotropic, motion-free basic state. The pressure is generally assumed to be hydrostatic and the fluid is assumed to be inviscid and adiabatic; the horizontal component of the Earth's rotation is neglected. This problem has been dealt with in great detail by Eckart [5], Margules [16], Dikii [4], Golitsyn and Dikii [7], Rossby et al. [22], Haurwitz [8], Longuet-Higgins [15], and many others. In view of the above approximations and assumptions, the solutions must be applied with caution to actual atmospheric phenomena. For some phenomena such as the daily tidal and thermotidal oscillations of the atmosphere the solutions provide a remarkably accurate description (Butler and Small [2], Lindzen [14]), while for other phenomena such as Rossby-Haurwitz waves the solutions provide an illuminating insight into the basic physics of a process which in its observed form is modified by baroclinity, nonlinearity, etc.

The problem, though simple in principle and conceptually important, is mathematically complicated. The equations it leads to are separable in latitude, longitude, and altitude dependences. However, the latitude dependence is described by Laplace's Tidal Equation, and as late as 1960, Eckart [5] could state that, "Despite the number of papers that have been devoted to this equation, its theory is still in a quite unsatisfactory state." Most of the solutions presently available for this equation apply either to specific cases or asymptotic limits (Hough

[11], Dikii [4], Golitsyn and Dikii [7], Lindzen [14], Kato [12], etc.). To a certain extent, even the recent, extensive numerical investigation of Laplace's Tidal Equation by Longuet-Higgins [15] suffers from these limitations. The difficulty of the equation has prevented the development of simple formulae of great generality.

In this paper we shall show that by the use of two β -planes—one centered at the equator, the other at some middle latitude—simple relations may be obtained which approximate with fair accuracy almost all results presently available from analyses of Laplace's Tidal Equation. The simplicity of the β -plane equations permits, without difficulty, the extension of our results to conditions on planets other than the Earth and to frequencies and wave numbers not previously explored in detail. In particular, new results will be presented on the vertical propagatability of planetary-scale waves with periods longer than a day. The present results will also give concrete examples of the adequacies and inadequacies of particular β -planes.

2. BASIC EQUATIONS

The equations are in essence those of classical atmospheric tidal theory as described in Siebert [23]. However, the spherical surface is replaced by a plane surface where y is distance in the northward direction and x is distance in the eastward direction. z is height above the surface. The only effect of the Earth's sphericity which is retained is to permit the vertical component of the Earth's rotation to vary linearly in y . This approximation is described in many places—Rossby et al. [22], and Veronis [26], [27],¹

¹ Veronis [27] runs into difficulty with the neglect of the horizontal component of the Earth's rotation. The mathematical foundations for this neglect are given by Phillips [19].

for example. The equations are, assuming time and longitude dependences of the form $e^{i(\omega t + kx)}$,

$$i\omega u - (f + \beta y)v = -ik \frac{\delta p}{\rho_0} \quad (1)$$

$$i\omega v + (f + \beta y)u = -\frac{1}{\rho_0} \frac{\partial}{\partial y} \delta p \quad (2)$$

$$\frac{\partial \delta p}{\partial z} = -g \delta \rho \quad (3)$$

$$i\omega \delta \rho + w \frac{d\rho_0}{dz} + \rho_0 \left(iku + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad (4)$$

$$i\omega \delta p + w \frac{dp_0}{dz} = \gamma g H \left(i\omega \delta \rho + w \frac{d\rho_0}{dz} \right) + (\gamma - 1) \rho_0 J \quad (5)$$

where

u = eastward velocity

v = northward velocity

w = vertical velocity

δp = pressure oscillation

g = acceleration by gravity

$\gamma = c_p/c_v = 1.4$

J = external heat excitation per unit mass per unit time

ρ_0 = basic density

p_0 = basic pressure

H = local scale height = RT_0/g

$f + \beta y$ is the approximation to $2\Omega \sin \theta$, where

Ω = Earth's rotation rate

θ = latitude

The origin of our y coordinate is that latitude, θ_0 , where $f = 2\Omega \sin \theta_0$; i.e., $f = 0$ if $\theta_0 = 0$. Similarly, $\beta = (2\Omega/a) \cos \theta_0$, where a = radius of Earth.

The basic state is taken to be motion free and dependent only on z . The basic fields are related by the following equations

$$\frac{\partial p_0}{\partial z} = -\rho_0 g \quad (6)$$

$$p_0 = \rho_0 RT_0 = \rho_0 g H \quad (7)$$

where T_0 = basic temperature.

Although consideration of a nonisothermal atmosphere is straightforward, we shall, in this paper, confine ourselves to an isothermal basic state. Then H is constant, and from (6) and (7)

$$\rho_0 = \rho_0(0) e^{-z/H} \quad (8)$$

The introduction of the following variables simplifies matters

$$v' = \rho_0^{1/2} v, \quad u' = \rho_0^{1/2} u, \quad w' = \rho_0^{1/2} w,$$

$$\delta p' = \rho_0^{-1/2} \delta p, \quad \delta \rho' = \rho_0^{-1/2} \delta \rho. \quad (9)$$

With (6)–(9), equations (1)–(5) become

$$i\omega u' - (f + \beta y)v' = -ik \delta p', \quad (10)$$

$$i\omega v' + (f + \beta y)u' = -\frac{\partial}{\partial y} \delta p', \quad (11)$$

$$\frac{\partial \delta p'}{\partial z} - \frac{1}{2H} \delta p' = -g \delta \rho', \quad (12)$$

$$i\omega \delta \rho' + \frac{\partial w'}{\partial z} - \frac{1}{2H} w' + iku' + \frac{\partial v'}{\partial y} = 0, \quad (13)$$

$$i\omega \delta p' = i\omega \gamma g H \delta \rho' + g(1 - \gamma)w' + (\gamma - 1)\rho_0^{1/2} J. \quad (14)$$

In passing, it should be noted, that the only effect of including a constant basic zonal velocity, u_0 , in our equations would be to replace ω in equations (10)–(14) by the Doppler shifted frequency $\omega + ku_0$. We shall return to this point later.

Eliminating u' , w' , and $\delta \rho'$ from equations (10)–(14) we obtain

$$((f + \beta y)^2 - \omega^2)v' = -i\omega \left(\frac{\partial}{\partial y} - \frac{k}{\omega} (f + \beta y) \right) \delta p', \quad (15)$$

and

$$\frac{\omega^2 H}{\kappa} \left[\frac{\partial^2 \delta p'}{\partial z^2} + \left(-\frac{1}{4H^2} + \frac{\kappa g}{H} \frac{k^2}{\omega^2} \right) \delta p' \right] + i\omega \left(\frac{\partial}{\partial z} - \frac{1}{2H} \right) (\rho_0^{1/2} J) + i\omega g \left(\frac{\partial v'}{\partial y} + \frac{k}{\omega} (f + \beta y)v' \right) = 0, \quad (16)$$

where $\kappa = (\gamma - 1)/\gamma$.

Eliminating $\delta p'$ from equations (15) and (16) leads to

$$\frac{H}{\kappa} \mathcal{L}_z [((f + \beta y)^2 - \omega^2)v'] + \left(\frac{\partial}{\partial y} - \frac{k}{\omega} (f + \beta y) \right) \left(\frac{\partial}{\partial z} - \frac{1}{2H} \right) (\rho_0^{1/2} J) + g \mathcal{M}_v [v'] = 0, \quad (17)$$

where

$$\mathcal{L}_z \equiv \frac{\partial^2}{\partial z^2} + \left(\frac{\kappa g}{H} \frac{k^2}{\omega^2} - \frac{1}{4H^2} \right) \quad (18)$$

$$\mathcal{M}_v \equiv \frac{\partial^2}{\partial y^2} + \frac{k}{\omega} \left(\beta - \frac{k}{\omega} (f + \beta y)^2 \right) \quad (19)$$

Now, for a given ω and k consider the set of functions $\{\Psi_{n,k,\omega}\}$ resulting from the following eigenfunction-eigenvalue equation

$$\mathcal{M}_v [\Psi_{n,k,\omega}] = \left(\frac{1}{gh_{n,k,\omega}} - \frac{k^2}{\omega^2} \right) ((f + \beta y)^2 - \omega^2) \Psi_{n,k,\omega}, \quad (20)$$

where $h_{n,k,\omega}$ (commonly called the equivalent depth) is the eigenvalue. Equation (20) is the β -plane counterpart

of Laplace's Tidal Equation (boundary conditions will be discussed later). Assuming $\{\Psi_{n,k,\omega}\}$ is complete, we may expand v' as follows:

$$v' = \sum_n V_{n,k,\omega}(z) \Psi_{n,k,\omega}(y) e^{i(kx + \omega t)}. \quad (21)$$

We may also write

$$\left(\frac{\partial}{\partial y} - \frac{k}{\omega}\right)(f + \beta y) \left(\frac{\partial}{\partial z} - \frac{1}{2H}\right)(\rho_0^{1/2} J) = ((f + \beta y)^2 - \omega^2) \sum_n S_{n,k,\omega}(z) \Psi_{n,k,\omega}(y) e^{i(kx + \omega t)}. \quad (22)$$

Equations (20), (21), (22), and (17) yield

$$\frac{d^2 V_n}{dz^2} + \left(\frac{\kappa}{Hh_n} - \frac{1}{4H^2}\right) V_n = -\frac{\kappa}{H} S_n, \quad (23)$$

where the subscripts k, ω are assumed to be understood. Equation (23) is merely the vertical structure equation of classical atmospheric tidal theory for an isothermal basic state. In the following two sections we will discuss equations (23) and (20) in greater detail.

3. VERTICAL STRUCTURE EQUATION

We will in the remainder of this paper consider two situations: (a) forced oscillations where $J \neq 0$, and k and ω are specified, and (b) free oscillations.

In the case of forced oscillations, the h_n values are obtained as eigenvalues of equation (20). The inhomogeneous equation (23) is then solved for the vertical structure of the various modes—subject to boundary conditions. The lower boundary condition is usually derived from the requirement $w=0$ at $z=0$. The upper boundary condition depends on the sign of the factor $[(\kappa/Hh_n) - (1/4H^2)]$ in equation (23). If it is negative, then the solutions behave exponentially in z , and the requirement that V_n remain bounded as $z \rightarrow \infty$ is sufficient. When $[(\kappa/Hh_n) - (1/4H^2)]$ is positive, the solutions are vertically propagating waves, and what the upper boundary condition should be is a matter of controversy although the radiation condition is often invoked (Wilkes [28], Yanowitch [29]). This controversy need not concern us here. The important point is merely that when $[(\kappa/Hh_n) - (1/4H^2)]$ is negative, energy is trapped near the levels of excitation, while when $[(\kappa/Hh_n) - (1/4H^2)]$ is positive, energy may propagate away from the levels of excitation.² This is shown in figure 1 for an atmosphere with $H=7.5$ km. ($T_0=256^\circ$ K.). For $h > 8.57$ km, the amplitude of v increases as one leaves the excitation levels (recall from (8) and (9) that

$$v = \frac{1}{\rho_0^{1/2}(0)} e^{z/2H} v'$$

however, the energy ($\alpha \rho_0 v^2$) decreases. For $h < 0$ the amplitude and energy both decrease. For $0 < h < 8.57$ energy propagates vertically in the form of waves with wave-

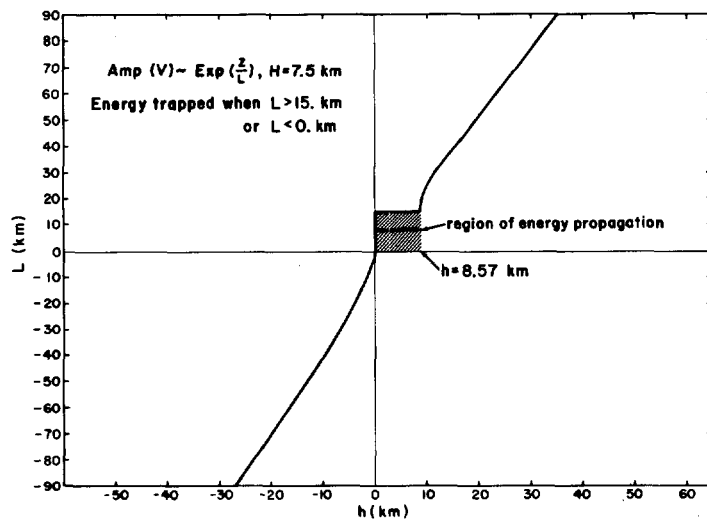


FIGURE 1.—Energy trapping as a function of equivalent depth (see text for details).

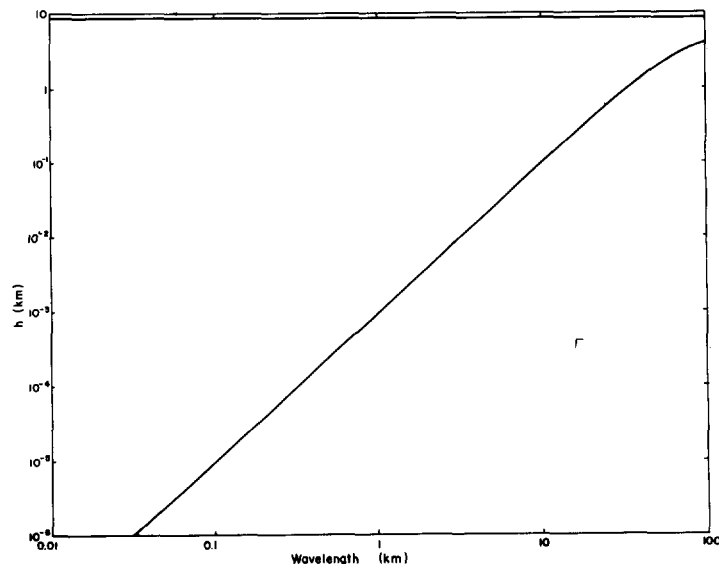


FIGURE 2.—Vertical wavelength as a function of equivalent depth.

lengths given by $2\pi/[(\kappa/Hh_n) - (1/4H^2)]^{1/2}$. Wavelength as a function of h is shown in figure 2.

In general, when $S_n=0$ in (23), the only solution satisfying both boundary conditions is $V_n=0$. However, there can exist values of h for which nontrivial homogeneous solutions exist. These values of h correspond to the free oscillations of the atmosphere. For an isothermal atmosphere where $J=0$, the condition $w=0$ at $z=0$ implies (after some manipulation) that

$$\frac{dv'}{dz} - \frac{\kappa}{H} \left(1 - \frac{1}{2\kappa}\right) v' = 0 \text{ at } z=0. \quad (24)$$

It turns out (Siebert [23], for example) that for an isothermal atmosphere there is only one h for which a homo-

² These and subsequent statements are for isothermal atmospheres. Exact analogous statements may be made for a thermally stratified atmosphere (Siebert [23], Dikii [4]).

geneous solution satisfying (24) and bounded as $z \rightarrow \infty$ exists. This h is given by

$$h = \gamma H. \quad (25)$$

For hypothetical temperature structures (in general unrealized in the real atmosphere) it is possible to have two and more h 's (Taylor [24], Wilkes [28]). However, the implication of the recent work of Fleagle [6] that there are an infinite number of free vertical modes is merely an artificial result of using a rigid lid at a finite altitude as an upper boundary. Our procedure for free modes will be to insert h , as given by equation (25), in (20) and consider ω rather than h as the eigenvalue. Alternately, we may consider those ω 's and k 's for which $h = \gamma H$ is an eigenvalue of equation (20) as the free modes of the atmosphere.

4. LATITUDE STRUCTURE EQUATION

Equation (20) may be rewritten

$$\frac{d^2 \Psi_{n,k,\omega}}{dy^2} + \left\{ \frac{1}{gh_{n,k,\omega}} (\omega^2 - f^2) + \frac{k}{\omega} \beta - k^2 - \frac{1}{gh_{n,k,\omega}} 2f\beta y - \frac{1}{gh_{n,k,\omega}} \beta^2 y^2 \right\} \Psi_{n,k,\omega} = 0. \quad (24)$$

Two special cases of equation (24) are usually studied: (a) an equatorially centered β -plane for which $f=0$, and $\beta=2\Omega/a$; (24) becomes

$$\frac{d^2 \Psi_{n,k,\omega}}{dy^2} + \left\{ \left(\frac{k}{\omega} \frac{2\Omega}{a} - k^2 + \frac{\omega^2}{gh_{n,k,\omega}} \right) - \left(\frac{2\Omega}{a} \right)^2 \frac{1}{gh_{n,k,\omega}} y^2 \right\} \Psi_{n,k,\omega} = 0, \quad (25)$$

(b) a midlatitude β -plane where f is evaluated at some middle latitude, and β is ignored unless it appears with constant factors; (24) becomes

$$\frac{d^2 \Psi_{n,k,\omega}}{dy^2} + \left\{ \frac{1}{gh} (\omega^2 - f^2) + k \left(\frac{\beta}{\omega} - k \right) \right\} \Psi_{n,k,\omega} = 0. \quad (26)$$

Applications of equation (25) in an oceanographic context may be found in Rattray [20], Hendershott [9], and Veronis [26], [27]; applications to the atmosphere may be found in Rosenthal [21] and Matsuno [17]. Applications of equation (26) are very widespread. Examples may be found in Rossby et al. [22], and Thompson [25].

Equation (26) is an approximation to (24); (25) is, in fact, identical to (24). The exact midlatitude version of (24) may be obtained from (25) by a shift of the y coordinate.³ It should also be noted that (25) can, fortuitously perhaps, be obtained from (26) by replacing f with $2\Omega y/a$ and β with $2\Omega/a$.

The following two intuitive points should be kept in mind when using (25) and (26):

(a) Because of the identity of equations (24) and (25), any solution of (25) which decays sufficiently fast before

$|y| = y_p$, the value y corresponding to the North Pole, does not depend on the fact that f , on a β -plane, goes to infinity and hence, is likely to be a valid approximation to solutions on a sphere. Any solution that does not decay before y_p cannot be a valid approximation.

(b) Because f , in equation (26), does not go to infinity, the solutions of (26) may be valid approximations in cases when the solutions of (25) are not.

5. GENERAL SOLUTIONS

EQUATORIALLY CENTERED β -PLANE

We shall take as our boundary conditions that $\Psi \rightarrow 0$ as $|y| \rightarrow \infty$. Let

$$c_1 = \left(\frac{2\Omega}{a} \right)^2 \frac{1}{gh_{n,k,\omega}} \quad (27)$$

$$c_2 = \frac{k}{\omega} \frac{2\Omega}{a} - k^2 + \frac{\omega^2}{gh_{n,k,\omega}} \quad (28)$$

$$\xi = c_1^{1/4} y. \quad (29)$$

Equation (25) becomes

$$\frac{d^2 \Psi_{n,k,\omega}}{d\xi^2} + (c_2 c_1^{-1/2} - \xi^2) \Psi_{n,k,\omega} = 0, \quad (30)$$

which is merely Schroedinger's equation for an harmonic oscillator whose solutions (Morse and Feshbach [18]) are given by

$$\Psi_{n,k,\omega} = e^{-1/2 \xi^2} H_n(\xi), \quad (31)$$

where

$$c_2 c_1^{-1/2} = \left(\frac{k}{\omega} \frac{2\Omega}{a} - k^2 + \frac{\omega^2}{gh_{n,k,\omega}} \right) \frac{\sqrt{gh_{n,k,\omega}}}{\left(\frac{2\Omega}{a} \right)} = 2n+1, \quad (32)$$

and $H_n(\xi)$ is the Hermite Polynomial of order n . Since we are dealing with an equatorially centered β -plane, the effective distance from the Earth's axis is a ; periodicity in the x -direction then requires that

$$k = \frac{s}{a}, \text{ where } s = 0, \pm 1, \pm 2, \dots \quad (33)$$

Equation (32) becomes

$$\left(\frac{s}{\omega} \frac{2\Omega}{a^2} - \frac{s^2}{a^2} + \frac{\omega^2}{gh_{n,s,\omega}} \right) \frac{\sqrt{gh_{n,s,\omega}}}{\left(\frac{2\Omega}{a} \right)} = 2n+1. \quad (34)$$

For free modes $h_{n,s,\omega}$ is replaced by $h = \gamma H$, and equation (34) is solved for ω , or more commonly for c , the longitudinal phase speed; i.e., $c = \omega a/s$. Equation (34) becomes

$$c^3 - \left(gh + (2n+1) \frac{2\Omega a}{s^2} \sqrt{gh} \right) c + \frac{2\Omega a}{s^2} gh = 0. \quad (35)$$

For forced modes, s and ω are given and (34) is solved for $h_{n,s,\omega}$ yielding

³ This, of course, is formally true only when f is linear in y .

$$\sqrt{gh_{n,s,\omega}} = \frac{a\Omega(2n+1)}{s^2 \left(\frac{2\Omega}{s\omega} - 1 \right)} \cdot \left[1 \pm \left(1 - \left(\frac{\omega}{\Omega} \right)^2 \left(\frac{s}{2n+1} \right) \left(\frac{2\Omega}{s\omega} - 1 \right) \right)^{1/2} \right]. \quad (36)$$

Note that equation (36) has two solutions. Recall from Section 4 that it is necessary for Ψ to decay before $|y|=y_p$ in order for Ψ to be a valid approximation to a solution on a sphere. Ψ begins to decay for $|y|$'s greater than that for which

$$\left(\frac{s}{\omega} \frac{2\Omega}{a^2} - \frac{s^2}{a^2} + \frac{\omega^2}{gh} \right) - \left(\frac{2\Omega}{a} \right)^2 \frac{1}{gh} y^2 = 0. \quad (37)$$

Let us denote this y by y_a . From (37), and (34)

$$y_a^2 = (2n+1) \frac{\sqrt{gh}}{\left(\frac{2\Omega}{a} \right)}. \quad (38)$$

y_a is thus seen to be smaller when the minus sign obtains in equation (36) than when the plus sign obtains. In practice, it has usually happened that Ψ is not a valid approximation when the plus sign obtains.

The condition $y_a < y_p$ is a necessary, but, as pointed out by Longuet-Higgins [15] and Matsuno [17], insufficient condition in one instance. When $n=0$, one solution of equation (35) is $c=2a\Omega$ and for this solution u' does not remain bounded as $y \rightarrow \infty$. Hence, the solution is invalid. Moreover it corresponds to no solution on a sphere.

MIDLATITUDE β -PLANE

The geometry for this case is shown in figure 3.

$$f = 2\Omega \sin \theta_0 \quad (39a)$$

$$\beta = \frac{2\Omega}{a} \cos \theta_0 \quad (39b)$$

$$d = a \left(\frac{\pi}{2} - \theta_0 \right) \quad (39c)$$

$$r = a \cos \theta_0. \quad (39d)$$

Also,

$$k = \frac{s}{a \cos \theta_0}, \quad s = 0, \pm 1, \pm 2, \dots, \quad (40)$$

as a result of longitudinal periodicity. The boundary conditions used are

$$\begin{aligned} \Psi &= 1 \text{ at } y = +d \text{ for } s=1, \\ \Psi &= 0 \text{ at } y = +d \text{ for } s \neq 1, \\ \Psi &= 0 \text{ at } y = -d. \end{aligned}$$

The conditions at $y = +d$ are based on the behavior of known Hough functions (i.e., solutions of Laplace's Tidal Equation). The condition $\Psi = 0$ at $y = -d$ is approximately relevant to symmetric heating functions. It should also be relevant to asymmetric heating functions when the Hough functions decay near the equator.

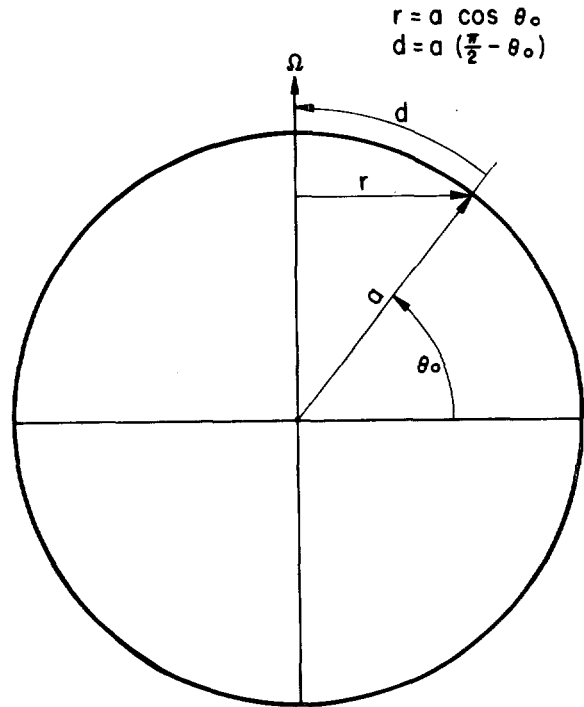


FIGURE 3.—Geometry for midlatitude β -plane (see text for details).

For the above boundary conditions the solutions to equation (26) are

$$\Psi_{n,s,\omega} = \sin \left(\frac{\pi n}{2d} (d-y) \right) \text{ for } s \neq 1 \quad (41a)$$

and

$$\Psi_{n,s,\omega} = \cos \left(\frac{(2n-1)\pi}{4d} (d-y) \right) \text{ for } s=1 \quad (41b)$$

where

$$\frac{1}{gh_{n,s,\omega}} (\omega^2 - 4\Omega^2 \sin^2 \theta_0) + \frac{s}{a \cos \theta_0} \left(\frac{2\Omega}{a} \cos \theta_0 - \frac{s}{a \cos \theta_0} \right) = \pi^2 \frac{(n-\delta_s)^2}{4d^2}, \quad (42)$$

and

$$\left. \begin{aligned} \delta_s &= 0 \text{ for } s \neq 1 \\ &= \frac{1}{2} \text{ for } s=1 \end{aligned} \right\} \quad (43)$$

For free modes (42) becomes

$$c^3 - \left\{ gh + \frac{a^2 \cos^2 \theta_0}{s^2} \left(4\Omega^2 \sin^2 \theta_0 + \frac{\pi^2 gh}{4d^2} (n-\delta_s)^2 \right) \right\} c + gh \frac{2\Omega a \cos^3 \theta_0}{s^2} = 0, \quad (44)$$

where

$$c = \frac{\omega a \cos \theta_0}{s}, \text{ and } h = \gamma H.$$

For forced modes we have from (42)

$$gh_{n,s,\omega} = \frac{\omega^2 - 4\Omega^2 \sin^2 \theta_0}{\frac{\pi^2}{4d^2} (n-\delta_s)^2 - \frac{2\Omega s}{a^2 \omega} + \frac{s^2}{a^2 \cos^2 \theta_0}}. \quad (45)$$

Note that for small enough ω and large enough s , $h_{n,s,\omega}$ is negative. Referring back to section 3 we see that negative h implies that the wave is trapped in the vertical. Note also that equation (45) has only one solution.

6. SOLAR DIURNAL OSCILLATIONS

The solar diurnal oscillation forms a particularly good case to check some of the equations developed in section 5. First, there are now available fairly complete solutions of Laplace's Tidal Equation for this case (Kato [12], Lindzen [14]). Second, these solutions fall naturally into two classes: one with eigenfunctions concentrated near the equator and positive equivalent depths, the other with eigenfunctions concentrated in middle latitudes and negative equivalent depths.

For the solar diurnal oscillations

$$\omega = \Omega \text{ and } s = 1. \quad (46)$$

Using equation (36) we obtain for the equatorial β -plane

$$\sqrt{gh_n} = a\Omega(2n+1) \left[1 \pm \left(1 - \frac{1}{(2n+1)^2} \right)^{1/2} \right], \quad (47)$$

and

$$\Psi_n = e^{-\frac{1}{2} \left(\frac{2\Omega}{a} \right) \frac{1}{\sqrt{gh_n}} y^2} H_n \left(\sqrt{\frac{2\Omega}{a}} \frac{1}{\sqrt{gh_n}} y \right), \quad (48)$$

where $n=0, 1, 2, \dots$. Even values of n correspond to asymmetric (with respect to the equator) thermal excitations, while odd values of n correspond to symmetric excitations. Equation (47) may be rewritten as follows

$$h_n = \frac{a^2 \Omega^2}{g} (2n+1)^2 \left[1 \pm \left(1 - \frac{1}{(2n+1)^2} \right)^{1/2} \right]^2, \quad (49)$$

where $a^2 \Omega^2 / g \cong 22.1$ km. for the earth. The values of h_n given by (49) are shown in table 1a.

In order for Ψ_n , as given by equation (48), to be valid y_a must be less than y_p (viz, sections 4 and 5). $y_p \cong \frac{\pi}{2} a$.

For $n \geq 1$, equation (49) may be approximated as follows

$$h_{-n} \cong \frac{1}{4} \frac{a^2 \Omega^2}{g} \frac{1}{(2n+1)^2} \quad (50)$$

$$h_{+n} \cong 4 \frac{a^2 \Omega^2}{g} (2n+1)^2 \quad (51)$$

From equations (50), (51), and (38) we have

$$y_a^2 \cong \frac{a^2}{4} \text{ for } h_{-n} \quad (52)$$

and

$$y_a^2 \cong (2n+1)^2 a^2 \text{ for } h_{+n}. \quad (53)$$

For $n=0$

$$y_a^2 = \frac{a^2}{2} \quad (54)$$

TABLE 1.—Equivalent depths:

a. for Solar Diurnal Equatorial Modes			
n	Equatorial β -plane		Kato [12]
	h_{-n} (km.)	h_{+n} (km.)	h_n "exact" (km.)
0	22.1	22.1	$+\infty$
1	.658	750.5	0.698
2	.221	2210.	.24
3	.1128	4332.	.121
4	.0682	7160.	.0722
5	.04565	10700.	.0489

b. for Solar Diurnal Midlatitude Modes			
n	Midlatitude β -plane	Kato [12]	
	h_n (km.)	h_n symmetric (km.)	h_n asymmetric (km.)
1	-10.4	-12.2	-----
2	-1.987	-1.75	-1.79
3	-.7595	-.63	-.64

for both h_{+0} and h_{-0} (they are identical). From equation (52) we see that all the solutions for h_{-n} should be valid approximations, while from (53) we see that all the solutions for h_{+n} (with the exception of $n=0$) should be invalid. On the other hand we see from (52) that those solutions which are valid span only the region between $y=0$ and $|y|=a/2$. There must, for purposes of completeness, also be solutions spanning the region between $a/2$ and $\pi a/2$. These are, presumably, described by the midlatitude β -plane.

Turning now to the midlatitude β -plane let us take $\theta_0 = \pi/3$. Then from equations (46) and (45) we obtain

$$h_n = -\frac{\Omega^2 a^2}{g} \frac{1}{\frac{9}{8} (2n-1)^2 + 1}. \quad (55)$$

Note that the h_n 's obtained from (55) are negative. They are listed for $n=1$ to 3 in table 1b.

Finally we list in tables 1a and b the "exact" values of h as obtained by Lindzen [14] and Kato [12]. We see the following:

(a) There is a consistent correspondence between midlatitude β -plane results and negative equivalent depths on a sphere. Moreover, the negative equivalent depths for symmetric and asymmetric modes on a sphere tend to become equal, while the β -plane doesn't distinguish one from the other. This property of negative equivalent depth modes has also been described by Longuet-Higgins [15].

(b) There is, not surprisingly, no correspondence between any of the "exact" values and the values of h_{+n} obtained from equation (49).

(c) For $n \geq 1$, there is a consistent correspondence between the values of h_{-n} obtained from (49) and the positive equivalent depths on a sphere.

(d) For $n=0$, $h_{-0} = 22.1$ km. This mode appears to correspond to the solution on a sphere for which $h = \infty$.

While the correspondence is not close, 22.1 km. is a large equivalent depth.

The results of this section show that for the solar diurnal oscillation, it is possible to approximate, by the use of two separate β -planes, almost all the results obtained from an analysis on a sphere. However, neither of the β -planes considered would have been adequate by itself.

7. ROSSBY-HAURWITZ WAVES

For the free oscillations of the atmosphere (where $h = \gamma H$ for an isothermal basic state) the relevant dispersion relations are either equation (35) for an equatorial β -plane or equation (44) for a midlatitude β -plane. Each is a cubic equation for c (or equivalently in ω). For sufficiently large s their three solutions may be interpreted as a gravity wave traveling west, a gravity wave traveling east, and an inertial oscillation (i.e., an oscillation for which $c \rightarrow 0$ as $\Omega \rightarrow 0$) traveling west. The first two correspond to Laplace's solutions of the first kind while the third corresponds to Laplace's solution of the second kind. In meteorology the third is known as a Rossby-Haurwitz wave. For small s , these waves are not readily distinguished from each other (viz, Matsuno [17], for example). It is not, however, the purpose of this paper to present a detailed analysis of equations (35) and (44). It will suffice to note that when the gravity and inertial modes are clearly separable, the dispersion relation for the latter is given by a balance of the last two terms in either (35) or (44). Thus, for the equatorial β -plane

$$c \doteq \frac{2\Omega a/s^2}{1 + (2n+1) \frac{2\Omega a}{s^2} \frac{1}{\sqrt{gh}}} \quad (56)$$

while for the midlatitude β -plane

$$c \doteq \frac{\frac{1}{\sqrt{2}} \frac{\Omega a}{s^2}}{1 + \frac{1}{s^2} \left(\frac{\Omega^2 a^2}{gh} + 2(n - \delta_s)^2 \right)}, \quad (57)$$

where θ_0 has been taken to be $\pi/4$. Equations (56) and (57) are quite similar. However, their dependences on $2\Omega a/\sqrt{gh}$ (called $\epsilon^{1/2}$ by Longuet-Higgins [15], and $\gamma^{1/2}$ by Golitsyn and Dikii [7]) are markedly different.

The dispersion relation on a sphere corresponding to (35) or (44) for β -planes is also cubic in c . Hence, for a given n and s , equations (56) and (57) must be different approximations to the same inertial wave. As an intuitive extension of the results of the last section we expect that (56) is a valid approximation when $y_a < y_p$. When $y_a > y_p$, we expect (57) will be a reasonable approximation. A comparison of equations (56) and (57) with asymptotic relations derived from Laplace's Tidal Equation will show these conjectures to be correct. We will demonstrate this at the end of this section.

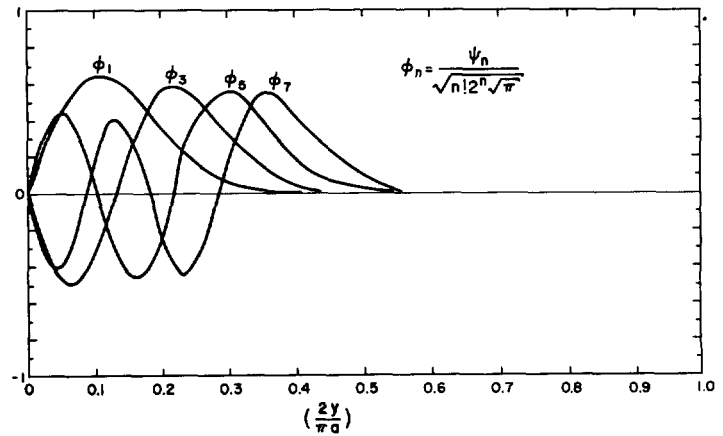


FIGURE 4.—Latitude distribution of meridional velocity of first few free Rossby-Haurwitz modes for $\epsilon = 4\Omega^2 a^2/gh = 1200$.

From (38) we have

$$\left(\frac{y_a^2}{a} \right) = (2n+1) \frac{1}{\epsilon^{1/2}}, \quad (58)$$

where

$$\epsilon = \frac{4\Omega^2 a^2}{gh}.$$

Equation (58) applies to all the solutions of (35)—not merely to (56); it shows that for ϵ sufficiently large and n sufficiently small, (56) ought to be the appropriate expression for the phase speed of a Rossby-Haurwitz wave. Equation (58) also shows that the limit $\epsilon \rightarrow \infty$ cannot be taken without regard to the behavior of n ; i.e., for a given ϵ , equation (56) should hold only if n is sufficiently small. This important point is ignored by Golitsyn and Dikii [7] and mentioned only peripherally by Longuet-Higgins [15].

We shall now see what ϵ is for various special cases. In the case of an isothermal basic state ($T_0 \cong 256^\circ \text{K.}$) for the Earth's atmosphere, $h = \gamma H \cong 10.5 \text{ km.}$ and $\epsilon \cong 11.9$; $\epsilon^{1/2} \cong 3.45$. Thus from equation (58), we have that the equatorial β -plane gives valid approximations for $n \leq 2$. For larger n 's, (57) would appear to be a more suitable approximation. For Jupiter, on the other hand, $\epsilon \cong 1200$ (Golitsyn and Dikii [7]) or $\epsilon^{1/2} \cong 34.6$. In this case (56) should be valid for $n \leq 20$. Only for larger n 's should (57) be appropriate on Jupiter (since (57) refers formally only to symmetric modes, n in (57) corresponds to $2n$ in (56)). The approximate eigenfunctions associated with (57) (viz, equation (41)) are sinusoidal and span all latitudes with comparable amplitudes. The latitude structures of the free oscillations of the Earth's atmosphere are more or less of this nature. For planets with large ϵ 's the eigenfunctions are not of this nature for sufficiently small n . In figure 4 we show the ψ_n 's (for the first few odd values of n) as given by equation (31) for $\epsilon = 1200$. Only as n becomes large do the eigenfunctions begin to span all latitudes. The structural implications of this difference for

the circulation of Jupiter's atmosphere have not yet been explored.

We finally come to the demonstration of the respective validity of equations (56) and (57) for small and large n . This demonstration is facilitated if we consider the non-dimensional period of the oscillations; i.e.,

$$\tau = \frac{\Omega}{ck} \quad (59)$$

instead of c . Equation (56) becomes

$$\tau \doteq \frac{s}{2} \left(1 + \frac{(2n+1)}{s^2} \epsilon^{1/2} \right), \quad (60)$$

or for large ϵ

$$\tau \doteq \frac{\epsilon^{1/2}}{s} \left(n + \frac{1}{2} \right) \quad (61)$$

Equation (57) becomes

$$\tau \doteq s \left[1 + \frac{1}{s^2} \left(\frac{\epsilon}{4} + 2(n - \delta_s)^2 \right) \right] \quad (62)$$

or for large ϵ and n

$$\tau \doteq \frac{s}{\epsilon} \left(\frac{\epsilon}{4} + 2n^2 \right). \quad (63)$$

Reference to Golitsyn and Dikii [7] now shows that (61) is identical with the asymptotic solution on a sphere for large ϵ (and sufficiently small n) while (63) is identical with Hough's asymptotic solution on a sphere for a given ϵ and sufficiently large n .⁴

8. FORCED OSCILLATIONS WITH PERIODS LONGER THAN A DAY

For oscillations of tidal periods the nature of the forcing is clear. For oscillations of other periods the nature (or even existence) of excitations is less clear. Conceptually, however, it proves convenient to assume that such features as nonlinearity and baroclinity in the basic state, omitted in equations (1)–(5), may excite a wide spectrum of disturbances in a manner analogous to the way in which turbulence may excite acoustic waves (Lighthill [13]). The consideration of long-period forced oscillations in the present context may, therefore, reveal features of at least qualitative relevance to the atmosphere's circulation. In particular, the differences between the equivalent depths (and hence vertical propagation properties) of modes characteristic of equatorial regions and of midlatitudes for the solar diurnal oscillation suggests that similar differences should exist for oscillations of longer periods as well.

In the light of the results of sections 6 and 7 we expect that the equivalent depths associated with the equatorial modes will be given approximately by equation (36) where the minus sign is assumed to obtain; i.e.,

$$h_{n,s,\omega}^{(\text{equatorial})} \doteq \left(\frac{a^2 \Omega^2}{g} \right) \left[\frac{2n+1}{s^2 \left(\frac{2\Omega}{s\omega} - 1 \right)} \right]^2 \cdot \left[1 - \left\{ 1 - \frac{\omega^2}{\Omega^2} \left(\frac{s}{2n+1} \right) \left(\frac{2\Omega}{s\omega} - 1 \right) \right\}^{1/2} \right]^2. \quad (64)$$

The eigenfunctions associated with equation (64) are given by (31). The latitude at which these modes begin to decay is given by (38). When

$$\frac{\omega^2}{\Omega^2} \left(\frac{s}{2n+1} \right)^2 \left(\frac{2\Omega}{s\omega} - 1 \right)$$

is sufficiently small, (64) and (38) yield

$$\left(\frac{y_a}{a} \right) \approx \frac{1}{2} \frac{\omega}{\Omega}. \quad (65)$$

Thus we see that the equatorial modes become increasingly confined to the equator as the period ($= 2\pi/\omega$) gets longer.

The equivalent depths associated with midlatitude modes should be given approximately by equation (45), which, for $\theta_0 = \pi/4$, may be rewritten

$$h_{n,s,\omega}^{(\text{midlatitude})} \doteq \left(\frac{a^2 \Omega^2}{g} \right) \frac{\left(\frac{\omega^2}{\Omega^2} - 2 \right)}{\left(4(n - \delta_s)^2 - 2 \frac{\Omega}{\omega} s + 2s^2 \right)}. \quad (66)$$

Both (64) and (66) are readily evaluated and extensive tables of solutions may be obtained in a negligible time on any current digital computer. For our purposes a consideration of selected cases will suffice.

We will consider the midlatitude modes first. In figure 5 we see h as a function of period ($= |2\pi/\omega|$) and longitudinal wave numbers ($|s|$) for $n=1$ and westward moving waves (ω and s both of the same sign). For periods of less than 5 days (increasing as s increases) h is either greater than 8.57 km. or negative and the waves cannot propagate vertically (see section 3). For longer periods h becomes positive and less than 8.57 km. For such periods waves can propagate vertically with wavelengths as given in figure 2. For periods less than 30 days, the vertical wavelengths are in excess of 10 km. The situation for values of n greater than 1 is similar, except that as n increases, the minimum period at which h changes from negative to positive occurs at increasing values of s . This is seen in figure 6 where the period at which h changes sign is shown as a function of n for different values of s . While the minimum period at which the change occurs corresponds to higher values of s as n increases, it is also true that for any particular s , the period for the changeover increases monotonically with n .

It is often remarked that tropospheric motions seem to decay with height as one reaches the vicinity of the tropopause because of the increasing static stability. Although high static stability inhibits buoyant convection, refer-

⁴ Golitsyn and Dikii [7] refer to Hough's solution as valid for small ϵ . Reference, however, to the original work of Hough [11] shows that it should be valid for any ϵ if n is sufficiently large.

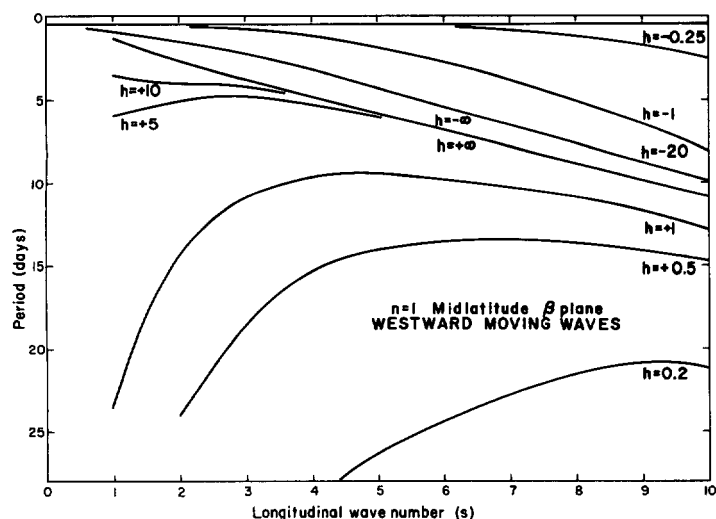


FIGURE 5.— h (in km.) as a function of period and longitudinal wave number for $n=1$ and westward moving waves (for midlatitude modes).

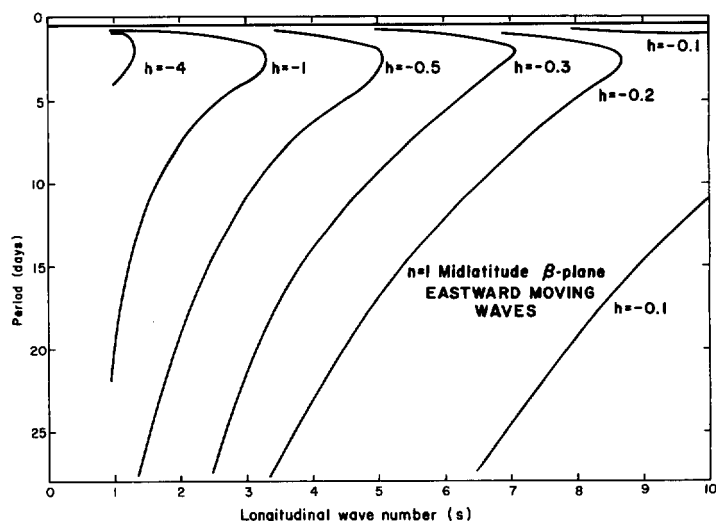


FIGURE 7.— h (in km.) as a function of period and longitudinal wave number for $n=1$ and eastward moving waves (for midlatitude modes).

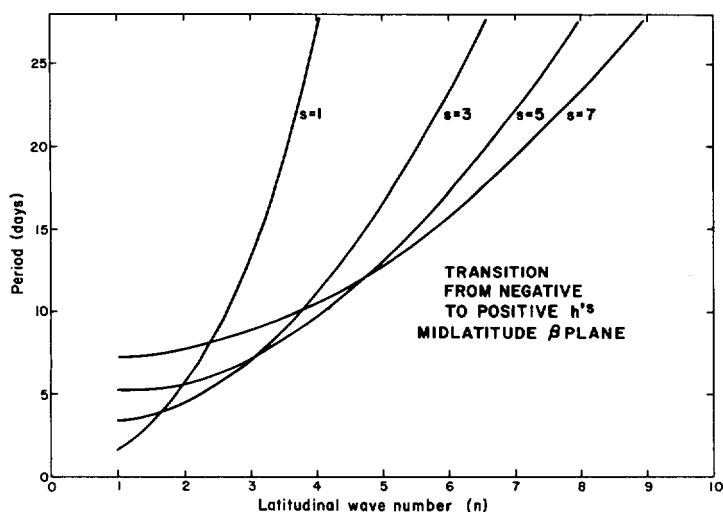


FIGURE 6.—Period at which h changes from negative to positive as a function of latitudinal wave number (for midlatitude modes).

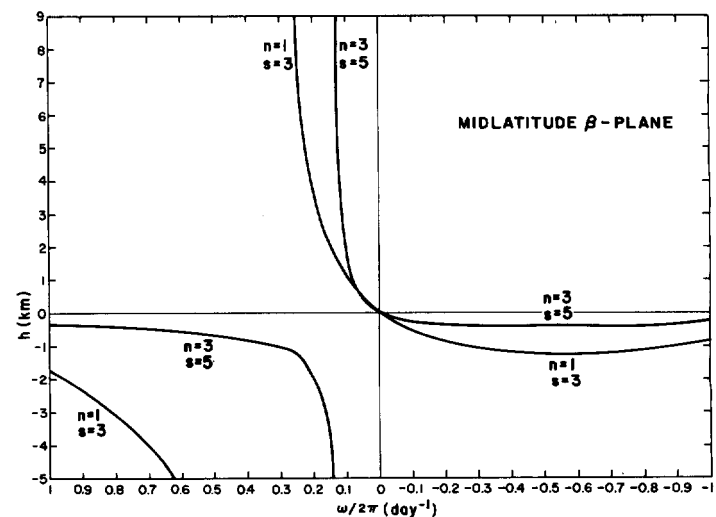


FIGURE 8.— h as a function of $\omega/2\pi$ ($=1/\text{period}$) for $n=1, s=3$ and $n=3, s=5$ (for midlatitude modes).

ence to Siebert [23] (where one may find the analog of equation (23) for a thermally stratified basic state) shows that static stability does not have such a role in trapping planetary waves. However, in figure 6 we see that for certain periods and wave numbers the Coriolis force due to the Earth's rotation prevents vertical communication. Since much of the energy of the atmosphere's motions (in the troposphere) occurs for periods in the neighborhood of 5 days and longitudinal wave numbers in the neighborhood of 5, this mechanism is likely to be of primary importance.

The situation for eastward moving waves (s and ω having different signs) is shown in figure 7 where h as a function of period and longitudinal wave number is shown

for $n=1$. In general, eastward moving waves are associated with small negative equivalent depths and are, therefore, strongly trapped. The situation for other values of n is similar.

One of the main obstacles to the applicability of the present results is that they have been developed for an atmosphere whose mean state is motion-free. What are the effects on planetary waves of a vertically varying mean zonal flow? From the work of Charney and Drazin [3], Bretherton [1], and Hines and Reddy [10] we see that the effects are in general complicated both to describe and to obtain. One of the effects, however, is to cause a Doppler shift in the frequency of a wave in a manner described in section 2, and this effect is readily describable

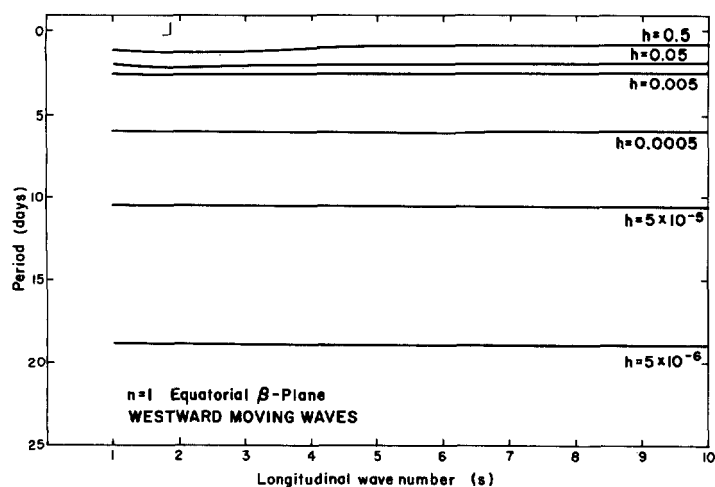


FIGURE 9.— h (in km.) as a function of period and longitudinal wave number for $n=1$ and westward moving waves (for equatorial modes).

in the present context. In figure 8 we show h as a function of $\omega/2\pi$ (i.e., $1/\text{period}$) for two particular pairs of n and s ($n=1, s=3$, and $n=3, s=5$). The particular choice of n, s is not significant. Let us consider a wave for which $n=3, s=5$ and $\omega/2\pi=0.5 \text{ day}^{-1}$ in the absence of any zonal flow. Such a wave will have a negative h and hence does not propagate vertically. An easterly zonal flow will, however, cause a Doppler shift in the frequency downward and for a sufficient easterly flow,

$$\frac{\omega \text{ Doppler shifted}}{2\pi} = \frac{(\omega + kU \text{ mean})}{2\pi}$$

will be less than 0.125 day^{-1} . From figure 8 we see that h will then be positive and less than 8.57 km . The wave may then propagate vertically. This result is similar to some results of Charney and Drazin [3] on the untrapping of planetary waves. Consider now a wave with $n=3, s=5$ and $\omega/2\pi=0.05 \text{ day}^{-1}$ in a layer of air with no mean zonal flow. Let this layer be surmounted by another layer with a large enough easterly flow to cause a Doppler shift in the frequency to a small negative value. The wave will propagate vertically in the lower layer, but will not propagate in the upper layer. Hence, it will be reflected at the interface. Because of diffusion, Kelvin-Helmholtz instabilities, etc., there can, of course, be no discontinuity in the zonal flow and the change must occur continuously. But, from figure 8 we see that as the Doppler shifted frequency approaches zero, h approaches zero. As a result, the vertical wavelength of the wave and hence its vertical phase speed will approach zero, and the wave will never reach the interface. These results are surprisingly similar to those obtained by Hines and Reddy [10] for gravity waves in a nonrotating atmosphere. The above hardly constitutes an analysis of wave propagation in the presence of

shear. It does, however, provide a framework for interpreting some of the results of more careful analyses.

Turning, now, to the equatorial modes, we find, not surprisingly, that the situation differs significantly from that at midlatitudes. In figure 9 we see h as a function of period and longitudinal wave number for $n=1$ and westward moving waves. The situation is essentially the same for other values of n (except $n=0$) and for eastward moving waves. h is small and positive for all cases considered. Thus, for all "long" periods and all longitudinal wave numbers the equatorial modes propagate vertically with short vertical wavelengths. The possibility exists that these equatorial modes might be excited by vertically trapped planetary waves at midlatitudes, and thus serve as an energy sink for midlatitude waves. The investigation of this possibility, while important, is beyond the scope of the present paper.⁵

9. CONCLUSIONS AND SUMMARY

The equations of classical atmospheric tidal theory were developed for an arbitrary β -plane. Two special β -planes were then considered: one centered at the equator, the other at a middle latitude. It was found that it is necessary and usually adequate to use both β -planes in order to approximate all the results that would be obtained from an analysis on a sphere. The analysis of the two β -planes is, however, far easier than the analysis of Laplace's Tidal Equation for a sphere. As examples of the utility of the two β -planes several separate examples were treated:

(a) Approximate formulae for the equivalent depths—both positive and negative—for solar diurnal oscillations were obtained.

(b) Dispersion relations for Rossby-Haurwitz waves on a rapidly rotating planet were obtained. In particular it was shown that the usual terrestrial formula holds for sufficiently large latitudinal wave numbers. For low latitudinal wave numbers the Rossby-Haurwitz waves are confined to tropical regions and are described by a somewhat different dispersion relation.

(c) Equivalent depths were obtained for terrestrial atmospheric oscillations of arbitrary longitudinal wave numbers and periods of one day or more. Associated with each longitudinal wave number and period were two sets of modes—one confined to equatorial regions, the other to the remaining latitudes. On interpreting equivalent depths as measures of the vertical propagation properties of the modes, it was found for the latter that all eastward traveling waves and all westward traveling waves with periods less than about 5 days are vertically trapped; westward propagating waves with sufficiently long periods can, however, propagate vertically. On the other hand, all the equatorial modes can propagate vertically—usually with very short vertical wavelengths.

⁵ It should be noted that the use of an upper lid in a numerical experiment would prevent the examination of the possibility.

ACKNOWLEDGMENTS

The author wishes to thank Drs. B. Haurwitz, W. Jones, and Mr. D. Moore for useful comments and discussions. Thanks are also due the reviewers for helpful corrections.

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[Received March 13, 1967; revised April 25, 1967]